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On surface-wave forcing by a circular disk

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The radiation resistance (damping coefficient) and virtual mass for a circular disk that executes small, heaving oscillations at the surface of a semi-infinite body of water, originally calculated by MacCamy (1961*a*) through the numerical solution of an integral equation, are calculated from a systematic hierarchy of variational approximations. The first member of this hierarchy is based on the exact solution of the boundary-value problem for $\alpha = 0$ and is in error by less than 2% for $0 \leqslant \alpha \leqslant 1$, where $\alpha = a\sigma^2/g$ (a = radius of disk, σ = angular frequency, g = gravity). The second approximation provides a variational interpolation between the limiting results for $\alpha = 0$ and $\alpha = \infty$ and appears to be in error by less than 2% for all α except in certain narrow intervals, where pseudoresonances pose difficulties. Those difficulties are overcome by local reference to the third approximation. Numerical results are plotted for $0 \leqslant \alpha \leqslant 10$. Asymptotic results for $\alpha \uparrow \infty$ are developed in an Appendix.

The corresponding formulation and the first variational approximation are developed for pitching oscillations of the disk.

1. Introduction

I consider here the excitation of gravity waves on the surface of the semi-infinite body of water $z > 0$ by a circular disk of radius a that executes small, heaving oscillations of angular frequency σ about the equilibrium position $z = 0$, $r < a$. This problem has been previously considered by MacCamy (1961*a*) and is closely related to Havelock's (1955) problem for a heaving sphere. Both disk and sphere may be regarded as idealized models of a ship or as laboratory wavemakers. The disk permits a simpler analytical formulation than the sphere, but at the expense of an edge singularity.

I assume that the fluid is incompressible and inviscid and that the motion originates from rest, by virtue of which there exists a complex velocity potential ϕ such that the particle velocity is given by

$$\boldsymbol{q} = \mathrm{Re}\{\boldsymbol{\nabla}\phi\, \mathrm{e}^{-\mathrm{i}\sigma t}\}. \tag{1.1}$$

The basic problem is to determine ϕ for a prescribed complex amplitude w of the velocity of the disk. The corresponding perturbation pressure, on the assumption of small disturbances, is given by

$$p - p_0 = \rho_0\, \mathrm{Re}\{\mathrm{i}\sigma\phi\, \mathrm{e}^{-\mathrm{i}\sigma t}\}, \tag{1.2}$$

where p_0 and ρ_0 are the ambient values of pressure and density. The complex amplitude of the vertical force on the disk is $\mathrm{i}\rho_0 \sigma\pi a^2 \langle\phi\rangle$, where, here and subsequently, $\langle\ \rangle$ signifies an average over the disk. The corresponding impedance, defined as the ratio of the complex amplitudes of force and velocity, is given by

$$Z = \mathrm{i}\rho_0 \sigma \pi a^2 \langle \phi \rangle / w \equiv R - \mathrm{i}\sigma M, \tag{1.3}$$

where R is the radiation resistance (the mean radiated power is $\frac{1}{2}R|w|^2$), and M is the virtual mass. It follows from dimensional considerations that

$$\hat{R} \equiv \frac{R}{(\rho_0 \sigma a^3)} \quad \text{and } \hat{M} \equiv \frac{M}{(\rho_0 a^3)} \tag{1.4a, b}$$

are functions of the single parameter

$$\alpha \equiv \frac{\sigma^2 a}{g} \equiv \kappa a. \tag{1.5}$$

MacCamy (1961a) uses Havelock's (1955) point-source Green function to obtain an integral equation that is equivalent to (A 1) below. He then reduces this integral equation to one with a somewhat simpler kernel, solves the reduced integral equation numerically, and determines the equivalents of R and M through numerical integration. This procedure fails for sufficiently large α and does not provide analytical approximations. I present here a variational formulation, following that for diffraction through a circular aperture (Levine & Schwinger 1948; Miles 1952), which begins (§3) with a Hankel transformation of the boundary-value problem and culminates in a variational form of Schwinger's type for the impedance Z. This formulation appears to be both simpler and more efficient than that of MacCamy, is useful for all α, and provides analytical approximations.

I first (in §4) substitute the limiting solution for $\alpha = 0$ directly into the variational form to obtain approximations to R and M that prove to be in error by less than 2% for $0 \leqslant \alpha \leqslant 1$. I then (in §5) develop a variational interpolation between the limiting results for $\alpha = 0$ and $\alpha = \infty$. The estimated error in this second approximation is, through comparison with a third approximation, less than 2% for all α except in certain narrow intervals, wherein pseudoresonances pose numerical difficulties. I circumvent these difficulties through local reference to the third approximation (which also exhibits pseudoresonances, but at different α). The integrals that appear in the variational approximations are Hilbert transforms, for which asymptotic approximations may be obtained through Ursell's (1983) method; I give the appropriate development in Appendix C.

The problems of radiation owing to pitching oscillations of, and scattering of a plane wave by, a circular disk, previously considered by Kim (1963) and MacCamy (1961b), respectively, admit formulations paralleling that of §§2–5. I sketch the formulation and develop the first variational approximation for pitching in §6.

2. Boundary-value problem

The assumption of incompressible, inviscid, irrotational flow implies

$$\nabla^2 \phi = 0 \quad (z > 0). \tag{2.1}$$

The linearized boundary condition on the free surface is

$$\phi_z + \kappa \phi = 0 \quad (z = 0,\ r > a), \tag{2.2a}$$

where $\kappa = \sigma^2/g$ is the wavenumber, whilst that on the disk is

$$\phi_z = w \quad (z = 0,\ r < a). \tag{2.2b}$$

In addition, ϕ must satisfy the null condition

$$\phi \to 0 \quad (z \uparrow \infty) \tag{2.3}$$

and the radiation condition

$$r^{\frac{1}{2}}(\phi_r - i\kappa\phi) \to 0 \quad (\kappa r \uparrow \infty, z = 0). \tag{2.4}$$

2.1. *The limit* $\alpha \downarrow 0$

The problem posed by (2.1)–(2.3) in the limit $\alpha \downarrow 0$, for which (2.2*a*) reduces to $\phi_z = 0$, reduces to that for a circular piston in a rigid baffle, for which the solution is given by (cf. Lamb 1932, §102, 2°)

$$\phi = -wa \int_0^\infty \frac{J_1(ka)\,J_0(kr)\,\mathrm{d}k}{k} = -\frac{2wa}{\pi} E\left(\frac{r}{a}\right) \quad (z = 0, r < a), \tag{2.5}$$

where E is a complete elliptic integral of the second kind. Averaging (2.5) over the disk and substituting the result into (1.3), we obtain

$$\hat{M} = \tfrac{8}{3} \quad (\alpha = 0). \tag{2.6a}$$

We infer from a calculation of the mean radiated power, using (3.9) and (4.4) below, that

$$\hat{R} \to \tfrac{1}{2}\pi^2\alpha \quad (\alpha \downarrow 0). \tag{2.6b}$$

2.2. *The limit* $\alpha \uparrow \infty$

The problem posed by (2.1)–(2.3) in the limit $\alpha \uparrow \infty$, for which (2.2*a*) reduces to $\phi = 0$, is equivalent to that for a circular disk moving along the z-axis in an infinite fluid. The potential on the disk is given by (Lamb 1932, §102, 4° and §108)

$$\phi = -\left(\frac{2w}{\pi}\right)(a^2 - r^2)^{\frac{1}{2}} \quad (z = 0, r < a), \tag{2.7}$$

the substitution of which into (1.3) yields

$$\hat{M} = \tfrac{4}{3} \quad (\alpha = \infty). \tag{2.8a}$$

We infer from (3.9) and (5.1) below that

$$\hat{R} \sim 8\alpha^{-1}\cos^2\alpha \quad (\alpha \uparrow \infty). \tag{2.8b}$$

We remark that the tangential velocities implied by (2.5) and (2.7) as $r \uparrow a$ on $z = 0+$ are singular like $-\log(a-r)$ and $(a-r)^{-\frac{1}{2}}$, respectively.

3. Integral-equation formulation

We begin the solution of (2.1)–(2.4) by introducing

$$f(r) \equiv (\phi_z + \kappa\phi)_{z=0}, \tag{3.1a}$$

which must vanish in $r > a$ in consequence of (2.2a) and reduces to

$$f = w + \kappa\phi, \tag{3.1b}$$

in $r < a$ by virtue of (2.2*b*). The solution then may be constructed through a Hankel transformation of the original problem and is given by

$$\phi(r, z) = -\int_0^\infty \frac{F(k)\,J_0(kr)\,\mathrm{e}^{-kz}\,k\,\mathrm{d}k}{k - \kappa}. \tag{3.2}$$

where the path of integration passes under the pole at $k = \kappa$ (see below), and

$$F(k) = \int_0^a f(r)\, J_0(kr)\, r\, \mathrm{d}r \tag{3.3}$$

is the Hankel transform of (3.1) and is implicitly determined by (2.2*b*), which yields the integral equation

$$\int_0^\infty \frac{F(k)\, J_0(kr)\, k^2\, \mathrm{d}k}{k-\kappa} = w \quad (r < a). \tag{3.4}$$

Equivalent integral equations for $f(r)$ are developed in Appendix A.

The asymptotic behaviour of (3.2) in the interior of the fluid, $r^2+z^2 \uparrow \infty$ with $r/z = O(1)$, is dominated by the contributions from the neighbourhood of $k = 0$ and exhibits the dipole behaviour

$$\phi \sim \frac{F(0)}{\kappa} \int_0^\infty J_0(kr)\, \mathrm{e}^{-kz}\, k\, \mathrm{d}k = \frac{F(0)}{\kappa} \frac{z}{(r^2+z^2)^{\frac{3}{2}}}. \tag{3.5}$$

We remark that this limit is not uniformly valid as $\kappa \downarrow 0$ and that the asymptotic behaviour for $\kappa = 0$, $\phi \sim -F(0)/(r^2+z^2)^{\frac{1}{2}}$, is source-like.

It remains to determine the radiated field and confirm the radiation condition (2.4). Substituting

$$J_0(kr) = \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} (\mathrm{e}^{\mathrm{i}kr\cos\theta} + \mathrm{e}^{-\mathrm{i}kr\cos\theta})\, \mathrm{d}\theta \tag{3.6}$$

into (3.2) and deforming the path of integration for the $\exp(\pm \mathrm{i}kr\cos\theta)$ component of the integrand to the positive/negative-imaginary k-axis, we find that the integral is dominated by the contribution of the pole in the limit $\kappa r \uparrow \infty$ with z fixed and may be approximated by

$$\phi \sim -2\mathrm{i}\kappa F(\kappa)\, \mathrm{e}^{-\kappa z} \int_0^{\frac{1}{2}\pi} \mathrm{e}^{\mathrm{i}\kappa r\cos\theta}\, \mathrm{d}\theta \tag{3.7a}$$

$$\sim \left(\frac{2\pi\kappa}{r}\right)^{\frac{1}{2}} F(\kappa)\, \mathrm{e}^{\mathrm{i}(\kappa r - \frac{3}{4}\pi) - \kappa z} \quad (\kappa r \uparrow \infty). \tag{3.7b}$$

[If the path of integration in (3.2) were indented over the pole at $k = \kappa$ (3.7*a*, *b*) would be replaced by their complex conjugates and would satisfy the radiation condition appropriate to an $\exp(\mathrm{i}\sigma t)$ time dependence.] The complex amplitude of the corresponding, free-surface displacement is

$$\zeta = \left(\frac{\mathrm{i}\sigma}{g}\right) \phi|_{z=0} \sim \kappa F(\kappa) \left(\frac{2\pi}{gr}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}(\kappa r - \frac{1}{4}\pi)}. \tag{3.8}$$

It follows from (3.8) (see Appendix B) that the mean radiated power is equal to $\frac{1}{2}R|w|^2$, as anticipated in §1. It also follows that

$$\hat{R} = 2\pi^2 \alpha \left|\frac{F(\kappa)}{a^2 w}\right|^2. \tag{3.9}$$

4. Variational formulation

Multiplying (3.4) through by $f(r)$, averaging over the disk, dividing the result through by $\langle f \rangle^2$, and invoking (3.3) and

$$\langle f \rangle = \left(\frac{2}{a^2}\right) \int_0^a f(r)\, r\, \mathrm{d}r = \left(\frac{2}{a^2}\right) F(0), \tag{4.1}$$

we obtain the variational form

$$\lambda \equiv \frac{w}{\langle f \rangle} = \tfrac{1}{2}a^2 \int_0^\infty \left[\frac{F(k)}{F(0)}\right]^2 \frac{k^2\, \mathrm{d}k}{k-\kappa}, \tag{4.2}$$

which is stationary with respect to first-order variations of $F(k)$ about the true-solution to (3.4) and invariant under a scale transformation of F (cf. Miles 1952).

This last result provides a direct approximation to the dimensionless impedance (1.4). Invoking (1.3) and (3.1 *b*), we obtain

$$\hat{R} - \mathrm{i}\hat{M} = \left(\frac{\pi}{\mathrm{i}\alpha}\right)(1-\lambda^{-1}). \tag{4.3}$$

It follows from (3.1 *b*) and (2.5) that $f = w[1+O(\alpha)]$ as $\alpha \downarrow 0$; accordingly, we expect the normalized ($\langle f_0 \rangle \equiv 1$) trial function

$$f_0 = 1, \quad F_0(k) = ak^{-1} J_1(ka), \tag{4.4 a, b}$$

to be suitable for moderate values of α. Substituting (4.4 *b*) into (4.2) and introducing $x = ka$ and $\alpha = \kappa a$, we obtain

$$\lambda_0 = 2 \int_0^\infty \frac{J_1^2(x)\, \mathrm{d}x}{x-\alpha}. \tag{4.5}$$

Separating out the contribution of the indentation under the pole at $x = \alpha$ and reducing the Cauchy principal value of the integral by invoking the integral representation of $J_1^2(x)$ [Watson 1945, §5.43 (2)] and then evaluating the resulting integral with respect to x as a Hilbert transform [Erdélyi *et al.* (1954), §15.3 (12)], we obtain

$$\lambda_0 = 2\mathrm{i}\pi J_1^2(\alpha) + 2\int_0^{\frac{1}{2}\pi} [H_0(2\alpha\cos\theta) + Y_0(2\alpha\cos\theta)]\cos 2\theta\, \mathrm{d}\theta \tag{4.6 a}$$

$$= 1 + \frac{8\alpha}{3\pi} + \tfrac{1}{2}\alpha^2\left[\ln\frac{2}{\alpha} - \gamma + \tfrac{1}{4} + \mathrm{i}\pi\right] - \frac{64}{45\pi}\alpha^3 + O(\alpha^4 \ln \alpha), \tag{4.6 b}$$

where H_0 is a Struve function and $\gamma = 0.5772\ldots$. Substituting (4.6 *b*) into (4.3), we obtain [cf. (2.6)]

$$\hat{R}_0 = \tfrac{1}{2}\pi^2\alpha\left(1 - \frac{16\alpha}{3\pi} + \ldots\right), \quad \hat{M}_0 = \frac{8}{3}\left[1 - \frac{3\pi}{16}\alpha(\ln\alpha + 1.075) + \ldots\right]. \tag{4.7 a, b}$$

It may be inferred from $f = w[1+O(\alpha)]$ and the variational principle that the $O(\alpha)$ term in (4.7 *a*) and the $O(1)$ and $O(\alpha \ln \alpha)$ terms in (4.7 *b*) are exact; cf. (2.6 *a*, *b*).

5. Variational interpolation

It follows from (2.7) and (3.1 *b*) that a suitable function for sufficiently large α is (after normalization to $\langle f_1 \rangle \equiv 1$)

$$f_1 = \frac{3}{2}\left(1 - \frac{r^2}{a^2}\right)^{\frac{1}{2}}, \quad F_1(k) = \frac{3}{2}\left(\frac{\pi a}{2k^3}\right)^{\frac{1}{2}} J_{\frac{3}{2}}(ka). \tag{5.1 a, b}$$

This, together with the results of the preceding section, suggests that a variational interpolation between (2.6) and (2.8) may be obtained through the trial function

$$F = A_0 F_0 + A_1 F_1. \tag{5.2}$$

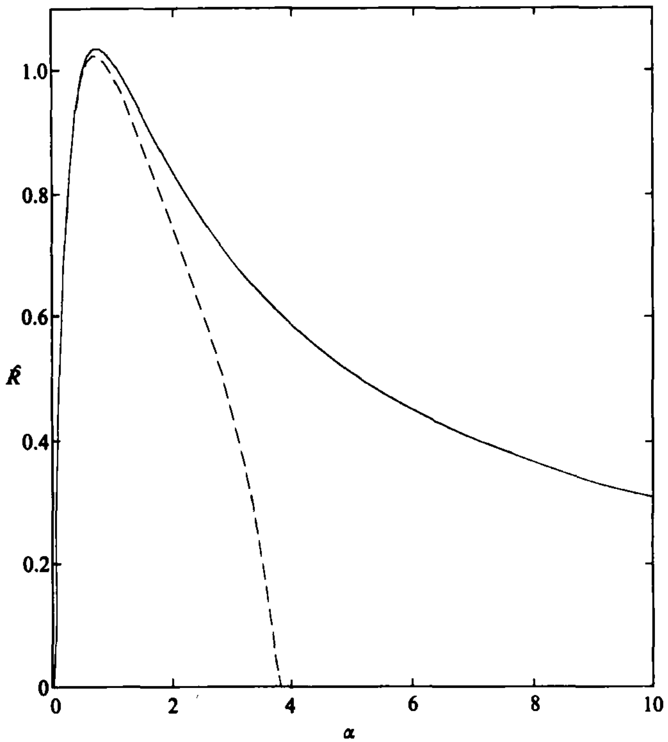

FIGURE 1. The dimensionless radiation resistance, $\hat{R} \equiv R/\rho_0 a^3\sigma$, as calculated from (4.3) using the approximations (4.5) (---) and (5.3) (——).

Substituting (5.2) into (4.2), choosing $A_0 \equiv 1$ (by virtue of the invariance of the variational form under a scale transformation of F), and invoking $\mathrm{d}\lambda/\mathrm{d}A_1 = 0$, we obtain

$$\lambda = \frac{\lambda_{00}\lambda_{11} - \lambda_{01}^2}{\lambda_{00} + \lambda_{11} - 2\lambda_{01}} \equiv \lambda_1, \tag{5.3}$$

where $\lambda_{00} \equiv \lambda_0$ (4.5),

$$\lambda_{11} = \frac{9\pi}{4}\int_0^\infty J_{\frac{3}{2}}^2(x)\frac{\mathrm{d}x}{x(x-\alpha)}, \quad \lambda_{01} = 3(\tfrac{1}{2}\pi)^{\frac{1}{2}}\int_0^\infty \frac{J_1(x)\,J_{\frac{3}{2}}(x)\,\mathrm{d}x}{x^{\frac{1}{2}}(x-\alpha)}. \tag{5.4a, b}$$

The approximation λ_{11} provides the correct leading terms in the asymptotic development of $\hat{R}$ and $\hat{M}$ (see Appendix C); however, it fails as $\alpha \downarrow 0$ (in which limit $\lambda_{11} \to \frac{9}{8}$ rather than 1) and is rather unsatisfactory for moderate α, which implies that f_0 is an essential component of the trial function f for finite α.

A systematic hierarchy of variational approximations to λ, of which λ_0 and λ_1 are the first and second members, may be obtained by expanding $f(r)$ in an appropriate, complete set of functions, of which f_0 and f_1 are the first two members; cf. Levine & Schwinger (1948), who use $f_n \propto [1-(r/a)^2]^{n-\frac{1}{2}}$ $(n = 1, 2, \ldots)$.

The required numerical integrations may be effected through the identity (for a path of integration indented under the pole at $x = \alpha$)

$$\int_0^\infty \frac{f(x)\,\mathrm{d}x}{x-\alpha} = \int_0^{2\alpha}\left[\frac{f(x)-f(\alpha)}{x-\alpha}\right]\mathrm{d}x + \int_{2\alpha}^\infty \frac{f(x)\,\mathrm{d}x}{x-\alpha} + \mathrm{i}\pi f(\alpha). \tag{5.5}$$

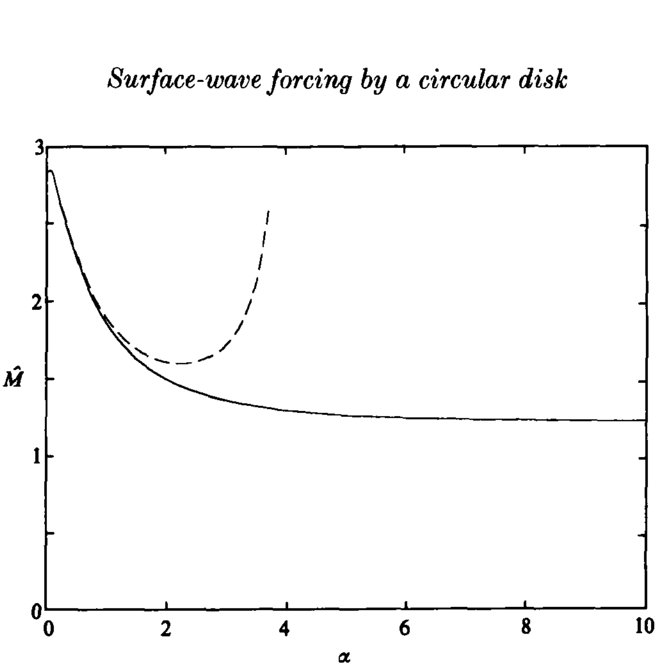

FIGURE 2. The dimensionless virtual mass, $\hat{M} \equiv M/\rho_0 a^3$, as calculated from (4.3) using the approximations (4.5) (---) and (5.3) (——). $\hat{M}$ achieves a maximum of 2.86 at $\alpha = 0.13$.

α	$\hat{R}$	$\hat{M}$
10	0.289	1.218
20	0.162	1.236
30	0.117	1.252
40	0.093	1.262
50	0.079	1.268
60	0.071	1.270
70	0.059†	1.251†
80	0.043	1.292
90	0.042	1.292
100	0.040	1.293

† These values are suspect owing to the proximity of $\alpha = 70$ to a pseudoresonance.

TABLE 1. Asymptotic approximations to $\hat{R}$ and $\hat{M}$ based on (C 11)–(C 13) and (5.3).

The approximations $\hat{R}_1$ and $\hat{M}_1$, obtained through the substitution of (5.3) into (4.3), are plotted in figures 1 and 2. These approximations are within 2% of the third approximations $\hat{R}_2$ and $\hat{M}_2$ except within certain narrow intervals (see below) in $0 \leqslant \alpha \leqslant 20$, where $\hat{R}_2$ and $\hat{M}_2$ are based on the incorporation of (the Hankel transform of) the additional degree of freedom $f_2 \propto [1-(r/a)^2]^{\frac{3}{2}}$ in (5.2). The approximations $\hat{R}_0$ and $\hat{M}_0$ (§4), which also are plotted in figures 1 and 2, differ from $\hat{R}_2$ and $\hat{M}_2$ by less than 2% for $0 \leqslant \alpha \leqslant 1$ but exhibit spurious oscillations for $\alpha \gtrsim 3$ in consequence of the corresponding oscillations of $J_1(\alpha)$ (the first zero of which is at $\alpha = 3.83$). These oscillations are almost completely smoothed out in $\hat{R}_1$ and $\hat{M}_1$, but the proximate zeros (*pseudoresonances*) of the numerator and denominator of (5.3) do pose difficulties in certain narrow intervals (e.g. $\alpha = 6.15 \pm 0.1$ and 9.6 ± 0.1). Similar pseudoresonances occur in $\hat{R}_2$ and $\hat{M}_2$, and presumably in higher approximations, but they are in different neighbourhoods, by virtue of which $\hat{R}_2$ and $\hat{M}_2$ may be used to smooth $\hat{R}_1$ and $\hat{M}_1$; this has been done in figures 1 and 2, although graphical interpolation would have sufficed. The smoothed approximations $\hat{R}_1$ and $\hat{M}_1$ agree with those of MacCamy (1961*a*), as presented by Kim (1965), within the accuracy ($\approx \pm 3\%$) of Kim's plots for $0 < \alpha < 4$. [The pseudoresonances, which are probably an intrinsic consequence of the modal expansion of $f(r)$, presumably move to increasingly large α as the level of truncation is increased. The difficulty is partly numerical in that the correct values of $\lambda_{00}\lambda_{11}-\lambda_{01}^2$ and $\lambda_{00}+\lambda_{11}-2\lambda_{01}$ and their higher-order counterparts may be smaller than, or comparable with, the errors in the numerical integration.]

The asymptotic expansions of λ_{00}, λ_{11} and λ_{01} for $\alpha \gg 1$ using Ursell's (1983) method for Hilbert transforms are carried out in Appendix C. The asymptotic approximations to $\hat{R}_1$ and $\hat{X}_1$ obtained through the substitution of (C11)–(C13) into (5.3) and (4.3) are given in table 1. It is evident that the approach to the asymptotes (2.8*a*, *b*) is extremely slow.

6. Pitching disk

If the circular disk executes a pitching oscillation about the axis $\theta = \frac{1}{2}\pi$, where θ is the azimuthal angle, the boundary condition (2.2*b*) is replaced by

$$\phi_z = -\Omega r \cos\theta \quad (z = 0,\, r < a), \tag{6.1}$$

where Ω is the maximum angular velocity of the disk. Proceeding as in §3, we pose the solution of (2.1), (2.2*a*), (2.3) and (2.4) in the form (cf. (3.2))

$$\phi(r, \theta, z) = -\cos\theta \int_0^\infty \frac{F_1(k)\, J_1(kr)\, \mathrm{e}^{-kz}\, k\, \mathrm{d}k}{k-\kappa}, \tag{6.2}$$

where the path of integration is indented under the pole at $k = \kappa$,

$$F_1(k) = \int_0^a f_1(r)\, J_1(kr)\, r\, \mathrm{d}r, \quad f_1(r)\cos\theta = (\phi_z + \kappa\phi)_{z=0}, \tag{6.3a, b}$$

and the subscript 1 now designates the azimuthal wavenumber. Invoking (6.1), we obtain the integral equation (cf. (3.4))

$$\int_0^\infty \frac{F_1(k)\, J_1(kr)\, k^2\, \mathrm{d}k}{k-\kappa} = -\Omega r \quad (r < a). \tag{6.4}$$

We define the counterpart of the complex impedance (1.3) as the ratio of the complex amplitude of the torque on the disk to the angular velocity Ω:

$$Z_1 \equiv R_1 - \mathrm{i}\sigma I = \frac{-\mathrm{i}\rho_0 \sigma \pi a^2 \langle \phi r \cos\theta \rangle}{\Omega} = \frac{\rho_0 \sigma \pi a^4}{4\mathrm{i}\kappa}\left(1 - \frac{1}{\lambda_1}\right), \quad \lambda_1 \equiv -\frac{\frac{1}{2}\Omega a^2}{\langle r f_1 \rangle}, \tag{6.5a, b}$$

where I is the virtual moment of inertia. Proceeding as in §4, we obtain the variational form (cf. (4.2))

$$\lambda_1 = (\tfrac{1}{2}a)^4 \int_0^\infty \left[\frac{F_1(k)}{F_1'(0)}\right]^2 \frac{k^2\, \mathrm{d}k}{k-\kappa}, \tag{6.6}$$

where

$$F_1'(0) \equiv \left(\frac{\mathrm{d}F_1}{\mathrm{d}k}\right)_{k=0} = \frac{1}{2}\int_0^a f_1(r)\, r^2\, \mathrm{d}r = \tfrac{1}{4}a^2 \langle r f_1 \rangle. \tag{6.7}$$

The limiting values of I may be determined by analogy with the procedures in §2 [in particular, the solution for $\alpha = \infty$ follows Lamb (1932), §109] and are given by

$$I \to \tfrac{4}{15}\rho_0 a^5 \quad (\alpha \downarrow 0), \quad I \to \tfrac{8}{45}\rho_0 a^5 \quad (\alpha \uparrow \infty). \tag{6.8a, b}$$

The coefficient $\frac{4}{15} = 0.2667\ldots$ compares with the value 0.266 determined by Kim (1963) through a numerical solution.

A variational approximation for $\alpha \lesssim 1$ may be obtained by positing the trial function $f_1 = r$ in (6.6). The end result is (cf. (4.5) and (4.6))

$$\lambda_1 = 4\int_0^\infty \frac{J_2^2(x)\,\mathrm{d}x}{x-\alpha}, \tag{6.9a}$$

$$= 4\mathrm{i}\pi J_2^2(\alpha) - 4\int_0^{\frac{1}{2}\pi} [H_0(2\alpha\cos\theta) + Y_0(2\alpha\cos\theta)]\cos 4\theta\,\mathrm{d}\theta, \tag{6.9b}$$

$$= \frac{\mathrm{i}\pi}{16}\alpha^4 + 1 + \frac{16\alpha}{15\pi} + \tfrac{1}{6}\alpha^2 + \frac{128\alpha^3}{315\pi} + \ldots. \tag{6.9c}$$

Substituting (6.9*c*) into (6.5), we obtain

$$\frac{Z_1}{\rho_0\sigma a^5} = \frac{\pi^2}{64}\alpha^3\left(1 - \frac{32\alpha}{15\pi} + \ldots\right) - \frac{4}{15}\mathrm{i}\left[1 + \left(\frac{5\pi}{32} - \frac{16}{15\pi}\right)\alpha + \left(\frac{1}{21} + \frac{256}{225\pi^2}\right)\alpha^2 + \ldots\right]. \tag{6.10}$$

The approximations to R_1 and I obtained through (6.5) and (6.9) agree with Kim's (1963) plots within the accuracy with which the plots can be read (± 2–3%) in $0 \leqslant \alpha \leqslant 2$.

Higher approximations may be obtained as in §5.

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Appendix A. Reduced integral equation

Combining (3.3) and (3.4), we obtain the equivalent integral equation

$$\int_0^a G(r,\rho) f(\rho)\,\rho\,\mathrm{d}\rho = w \quad (r < a), \tag{A 1}$$

where

$$G(r,\rho) = \int_0^\infty \frac{J_0(kr)\,J_0(k\rho)\,k^2\,\mathrm{d}k}{k-\kappa}. \tag{A 2}$$

Invoking the identities

$$\frac{k^2}{k-\kappa} = k + \kappa + \frac{\kappa^2}{k-\kappa}, \tag{A 3}$$

$$(\partial_r^2 + r^{-1}\partial_r)\,J_0(kr) \equiv \Delta J_0(kr) = -k^2 J_0(kr), \tag{A 4}$$

and the inverse transforms

$$\int_0^\infty J_0(kr)\,J_0(k\rho)\,k\,\mathrm{d}k = \frac{\delta(r-\rho)}{\rho}, \tag{A 5}$$

$$\int_0^\infty J_0(kr)\,J_0(k\rho)\,\mathrm{d}k = \frac{(2/\pi)}{r+\rho}K\left[\frac{2(r\rho)^{\frac{1}{2}}}{r+\rho}\right] \equiv G_1(r,\rho), \tag{A 6}$$

where δ is Dirac's delta function, and K is an elliptic integral of the first kind, we obtain

$$G(r,\rho) = (1+\kappa^2\Delta^{-1})^{-1}\left[\frac{\delta(r-\rho)}{\rho}+\kappa G_1(r,\rho)\right]. \tag{A 7}$$

Substituting (A 7) into (A 1) and multiplying the result by the operator $1+\kappa^2\Delta^{-1}$, we obtain the reduced integral equation,

$$f(r)+\kappa\int_0^a G_1(r,\rho)f(\rho)\,\rho\,\mathrm{d}\rho = (1+\kappa^2\Delta^{-1})\,w \quad (0 \leqslant r < a), \tag{A 8}$$

which is equivalent to MacCamy's (1961*a*) equation (52) after restoring a missing factor of $(1/2\pi)$ therein.

Appendix B. Radiated power

The mean surface-wave energy, which is half potential and half kinetic, is $\frac{1}{2}\rho_0 g|\zeta|^2$ per unit area. Invoking (3.8) for ζ and $c_g = \frac{1}{2}(\sigma/\kappa)$ for the group velocity, we then have

$$P = c_g(2\pi r)\,(\tfrac{1}{2}\rho_0 g|\zeta|^2) = \pi^2\rho_0\,\sigma\kappa|F(\kappa)|^2 \tag{B 1}$$

for the mean radiated power.

Multiplying (3.4) through by the complex conjugate $\bar{f}(r)$, averaging over the disk, taking the imaginary part of the result (which is derived entirely from the indentation of the path of integration under the pole at $k=\kappa$), and substituting $\langle\bar{f}\rangle = \bar{w}/\bar{\lambda}$ from (4.2), we obtain

$$\pi\kappa^2|F(\kappa)|^2 = \tfrac{1}{2}a^2\,\mathrm{Im}\,(w\langle\bar{f}\rangle) = \tfrac{1}{2}a^2|\lambda|^{-2}\lambda_i|w|^2. \tag{B 2}$$

Substituting (B 2) into (B 1) and eliminating $|\lambda|^{-2}\lambda_i$ through (4.3), we obtain

$$P = \tfrac{1}{2}R|w|^2. \tag{B 3}$$

Appendix C. The limit $\alpha\uparrow\infty$

The asymptotic expansion of the Hilbert transform

$$\mathscr{H}\{f(x);\alpha\} = ⨍_0^\infty \frac{f(x)\,\mathrm{d}x}{x-\alpha}, \tag{C 1}$$

when f is an analytic function of x that admits an asymptotic expansion of the form

$$f(x) \sim \sum_{n=0}^{\infty} x^{-n-\nu}(a_n + A_n\cos\omega x + B_n\sin\omega x) \quad (0<\nu\leqslant 1) \tag{C 2}$$

has been developed by Ursell (1983). Applying his results to the integral

$$\lambda = \int_0^\infty \frac{f(x)\,\mathrm{d}x}{x-\alpha} = \mathscr{H}\{f(x);\alpha\} + \mathrm{i}\pi f(\alpha), \tag{C 3}$$

we obtain

$$\lambda \sim -\sum_{n=0}^{\infty}\mathscr{M}(n+1)\,\alpha^{-n-1} + \mathrm{i}\pi\sum_{n=0}^{\infty}\left[a_n(1-\mathrm{i}\cot\pi\nu)+(A_n-\mathrm{i}B_n)\,\mathrm{e}^{\mathrm{i}\omega\alpha}\right]\alpha^{-n-\nu} \tag{C 4}$$

for $0 < \nu < 1$ or

$$\lambda \sim \sum_{n=0}^{\infty} [a_n(\mathrm{i}\pi - \ln\alpha) - d_n + \mathrm{i}\pi(A_n - \mathrm{i}B_n)\,\mathrm{e}^{\mathrm{i}\omega\alpha}]\,\alpha^{-n-1} \tag{C 5}$$

for $\nu = 1$ (for which the present notation differs slightly from that of Ursell), where

$$\mathscr{M}(\rho) \equiv \mathscr{M}\{f(x);\rho\} = \int_0^{\infty} f(x)\,x^{\rho-1}\,\mathrm{d}x \tag{C 6}$$

is the Mellin transform of f, and

$$d_n = \lim_{\rho\to n+1}\left[\mathscr{M}(\rho) + \frac{a_n}{\rho-(n+1)}\right]. \tag{C 7}$$

This last result reduces to $d_n = \mathscr{M}(n+1)$ if $a_n = 0$.

The result (C 5) provides the asymptotic expansions of λ_{00} and λ_{11}, for which

$$f_{00}(x) = 2J_1^2(x) = \frac{2}{\pi}\left[\frac{1-\sin 2x}{x} - \frac{3}{4}\frac{\cos 2x}{x^2} + \frac{3}{8}\left(\frac{1-\frac{1}{4}\sin 2x}{x^3}\right) + \frac{15}{128}\frac{\cos 2x}{x^4} + O(x^{-5})\right], \tag{C 8}$$

$$f_{11}(x) = \tfrac{9}{4}\pi x^{-1} J_{\frac{3}{2}}^2(x) = \frac{9}{4}\left[\frac{1+\cos 2x}{x^2} - \frac{2\sin 2x}{x^3} + \frac{1-\cos 2x}{x^4}\right], \tag{C 9}$$

whilst (C 4) with $\nu = \frac{1}{2}$ provides the expansion of λ_{01}, for which

$$f_{01}(x) = 3\left(\frac{\pi}{2x}\right)^{\frac{1}{2}} J_1(x)\,J_{\frac{3}{2}}(x) = \tfrac{3}{2}\pi^{-\frac{1}{2}}\left[\frac{1+\cos 2x - \sin 2x}{x^{\frac{3}{2}}}\right.$$
$$\left. + \frac{5-11\cos 2x - 11\sin 2x}{8x^{\frac{5}{2}}} + \frac{63-33\cos 2x + 33\sin 2x}{128x^{\frac{7}{2}}} + O(x^{-\frac{9}{2}})\right]. \tag{C 10}$$

The required Mellin transforms are given by §6.8 (33) in Erdélyi *et al.* (1954). The end results are

$$\lambda_{00} = \frac{2}{\pi}[\pi(\mathrm{i}-\mathrm{e}^{2\mathrm{i}\alpha}) + 2 - \gamma - \ln 8\alpha]\,\alpha^{-1} - \tfrac{3}{2}\,\mathrm{i}\,\mathrm{e}^{2\mathrm{i}\alpha}\,\alpha^{-2}$$
$$+ \frac{3}{4\pi}\left[\pi(\mathrm{i}-\tfrac{1}{4}\,\mathrm{e}^{2\mathrm{i}\alpha}) + \tfrac{11}{6} - \gamma - \ln 8\alpha\right]\alpha^{-3} + \frac{15\mathrm{i}}{64}\,\mathrm{e}^{2\mathrm{i}\alpha}\,\alpha^{-4} + O(\alpha^{-5}\ln\alpha), \tag{C 11}$$

$$\lambda_{11} = -\frac{3\pi}{4}\alpha^{-1} + \tfrac{9}{4}[\mathrm{i}\pi(1+\mathrm{e}^{2\mathrm{i}\alpha}) + 1 - \gamma - \ln 2\alpha]\,\alpha^{-2} - \tfrac{9}{2}\pi\,\mathrm{e}^{2\mathrm{i}\alpha}\,\alpha^{-3}$$
$$+ \tfrac{9}{4}[\mathrm{i}\pi(1-\mathrm{e}^{2\mathrm{i}\alpha}) + \tfrac{5}{4} - \gamma - \ln 2\alpha]\,\alpha^{-4} + O(\alpha^{-5}), \tag{C 12}$$

$$\lambda_{01} = -\frac{3\pi}{4}\alpha^{-1} + \tfrac{3}{2}\mathrm{i}\pi^{\frac{1}{2}}[1+(1+\mathrm{i})\,\mathrm{e}^{2\mathrm{i}\alpha}]\,\alpha^{-\frac{3}{2}} + \frac{3\mathrm{i}\pi^{\frac{1}{2}}}{16}[5-11\,(1-\mathrm{i})\,\mathrm{e}^{2\mathrm{i}\alpha}]\,\alpha^{-\frac{5}{2}}$$
$$+ \frac{9\mathrm{i}\pi^{\frac{1}{2}}}{256}[21-11(1+\mathrm{i})\,\mathrm{e}^{2\mathrm{i}\alpha}]\,\alpha^{-\frac{7}{2}} + O(\alpha^{-\frac{9}{2}}), \tag{C 13}$$

where $\gamma = 0.577215\ldots$ is Euler's constant.

Numerical results obtained through the substitution of (C 11)–(C 13) into (5.3) and (4.3) are given in table 1. These results exhibit pseudoresonances similar to those in the results based on numerical integration; see discussion following (5.5). The substitution of (C 12) into (4.3) yields

$$\tilde{R}_{11} \sim 8\alpha^{-1}\cos^2\alpha, \quad \tilde{M}_{11} \sim \tfrac{4}{3}-\alpha^{-1}\left\{\frac{4}{\pi}(\ln 2\alpha+\gamma-1)+4\sin 2\alpha-\pi\right\}, \qquad \text{(C 14}a, b\text{)}$$

which are asymptotically correct by virtue of the variational principle and the asymptotic validity of f_1, but are rather unsatisfactory approximations in the range of physical interest.

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